Intergrerend project dynamische systemen

H. Waalkens 2009

university of groningen

Exam Problem Sheet

The exam consists of 5 problems. You may answer in Dutch or in English. You can achieve 38 points in total.

1. [**3+3+3 Points.**]
   Consider the family of differential equations
   $$x' = ax + \sin x\,,$$
   where a is a parameter.
   (a) Sketch the phase line when $a = 0$.
   (b) Use the graphs of ax and $\sin x$ to determine the qualitative behavior of all the bifurcations that occur as a increases from -1 to 1.
   (c) Sketch the bifurcation diagram for this family of differential equations.
2. [**3+2+4 Points.**]
   Consider the differential equation $x' = x + \cos t$.
   (a) Find the general solution of this equation.
   (b) Prove that there is a unique periodic solution for this equation.
   (c) Compute the Poincaré map $p : t = 0 \to t = 2\pi$ for this equation and use this to verify again that there is a unique periodic solution.
3. [**5 Points.**]
   Consider the harmonic oscillator equation (with mass $m = 1$)
   $$x'' + bx' + kx = 0\,,$$
   where $b \geq 0$ and $k > 0$. Identify the regions in the relevant portion of the $b-k$ plane where the corresponding system has similar phase portraits.
4. [**4 Points.**]
   Consider the first order differential equation
   $$x' = f_a(x)$$
   for which $f_a(x_0) = 0$ and $f_a'(x_0) \neq 0$. Prove that the differential equation
   $$x' = f_{a+\epsilon}(x)$$

has an equilibrium point $x_0(\epsilon)$ where for ϵ sufficiently small, $\epsilon \mapsto x_0(\epsilon)$ is a smooth function satisfying $x_0(0) = x_0$.

5. [**3+2+2+4 Points.**]

   (a) Give the definition of stability and asymptotic stability for equilibrium points.

   (b) Give the definition of an hyperbolic equilibrium point. What can one say about the stability of an hyperbolic equilibrium point?

   (c) How is a saddle equilibrium point of a planar system defined? What is its canonical form?

   (d) State the stable curve theorem for saddle equilibrium points of planar systems.